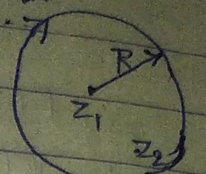
(Defn) **Integral function:-** A function $f(z)$ which is analytic in every finite region of z plan is known as integral function

QNo.→ State and Prove Liouville's theorem.

Ans:- **Statement:-** Let $f(z)$ be an integral function satisfying the inequality $|f(z)| \leq M$ for all values of z where M is a +ve constant. then $f(z)$ is constant.

Let z_1, z_2 be any two points in the z-plane. γ be a circle with centre z_1 & radius R such that the point z_2 is interior to γ.



Now, by Cauchy's integral formula, we have

$$f(z_1) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_1}\,dz$$

and $$f(z_2) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_2}\,dz.$$

$$\therefore f(z_2) - f(z_1) = \frac{1}{2\pi i}\int_\gamma \left(\frac{1}{z-z_2} - \frac{1}{z-z_1}\right) f(z)\,dz$$

$$= \frac{z_2 - z_1}{2\pi i}\int_\gamma \frac{f(z)\,dz}{(z-z_1)(z-z_2)} \quad \text{——} ①$$

We now choose R ~~that~~ so large that $|z_2 - z_1| < \frac{R}{2}$.

Since, $|z - z_1| = R$, we have

$$|z - z_2| = |z - z_1 + z_1 - z_2| = |(z-z_1) - (z_2 - z_1)|$$

$$\geq |z - z_1| - |z_2 - z_1| \geq R - \frac{R}{2} = \frac{R}{2}.$$

It is given that $|f(z)| \leq M$, therefore from ①, we have

$$|f(z_2) - f(z_1)| = \left|\frac{z_2 - z_1}{2\pi i}\int_\gamma \frac{f(z)\,dz}{(z-z_1)(z-z_2)}\right|$$

$$\leq \frac{|z_2 - z_1|}{2\pi}\int_\gamma \frac{|f(z)|\,|dz|}{|z-z_1|\,|z-z_2|}$$

$$\leq \frac{M|z_2 - z_1|}{\cancel{2\pi R} \cdot \frac{R}{2}} \cdot \cancel{2\pi R} = \frac{2M|z_2 - z_1|}{R} \to 0 \text{ as } R \to \infty$$

$\therefore f(z_2) - f(z_1) = 0$

$\therefore f(z_2) = f(z_1)$

As z_1 & z_2 are arbitrary it follows that $f(z)$ is Constt.